EXISTENCE OF PARABOLIC BOUNDARY POINTS OF CERTAIN DOMAINS IN $\mathbb{C}^2$

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ABSTRACT. In this paper, the existence of parabolic boundary points of certain convex domains in $\mathbb{C}^2$ is given. On the other hand, the nonexistence of parabolic boundary points of infinite type of certain domains in $\mathbb{C}^2$ is also shown.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$. Denote by $Aut(\Omega)$ the group of holomorphic automorphisms of Ω. The group $Aut(\Omega)$ is a topological group with the natural topology of uniform convergence on compact sets of Ω (i.e., the compact-open topology).

It is a standard and classical result of H. Cartan that if Ω is a bounded domain in $\mathbb{C}^n$ and the automorphism group of Ω is noncompact then there exist a point $x \in \Omega$, a point $p \in \partial\Omega$, and automorphisms $\varphi_j \in Aut(\Omega)$ such that $\varphi_j(x) \to p$. In this circumstance we call p a *boundary orbit accumulation point*. The classification of domains with noncompact automorphism group relies deeply on the study the geometry of the boundary at an orbit accumulation point p. For instance, Wong and Rosay [15],[16] showed that if p is a strongly pseudoconvex point, then the domain is biholomorphic to the ball. In [1], [2], [3]and [5], E. Bedford, S. Pinchuk and F. Berteloot showed that if p is of finite type, then the domain is biholomorphic to the domain of the following form

$$M_P = \{(w, z) \in \mathbb{C}^2 : \mathrm{Re}w + P(z, \bar{z}) < 0\}$$

where P is an homogeneous polynomial in z and $\bar{z}$. Each domain M_P is called a model of Ω at p. To prove this, they first applied the Scaling method to point out that $Aut(\Omega)$ contains a parabolic subgroup, i.e.,

there is a point $p_\infty \in \partial\Omega$ and a one-parameter subgroup $\{h^t\}_{t\in\mathbb{R}} \subset Aut(\Omega)$ such that for all $z \in \Omega$

$$\lim_{t\to\pm\infty} h^t(z) = p_\infty. \tag{1.1}$$

Each boundary point satisfying (1.1) is called a parabolic boundary point of Ω. After that, the local analysis of a holomorphic vector field H which generates the above subgroup h^t was carried out to show that Ω is biholomorphic to the desired homogeneous model.

We now consider a bounded domain $\Omega \subset \mathbb{C}^2$. Suppose that Ω is biholomorphic to the domain D defined by $D = \{(w,z) \in \mathbb{C}^2 : \mathrm{Re}w + \sigma(z) < 0\}$ with some smooth real-valued function on the complex plane. The one-parameter of translations $\{L^t\}_{t\in\mathbb{R}}$ given by $L^t(w,z) = (w + it, z)$ acts on the domain D. The transformation $\psi : D \to \Omega$ allows us to define the one-parameter group of biholomorphic mappings $\{h^t := \psi^{-1} \circ L^t \circ \psi\}_{t\in\mathbb{R}}$ acting on Ω. The fisrt aim of this paper is to show that this one-parameter group is parabolic. Namely, we prove the following theorem.

Theorem 1.1. *Let Ω be a $\mathcal{C}^1$-smooth, bounded, strictly geometrically convex domain in $\mathbb{C}^2$. Let $\psi : \Omega \to D$ be a biholomorphism, where $D := \{(w,z) \in \mathbb{C}^2 : Rew + \sigma(z) < 0\}$, σ is a $\mathcal{C}^1$-smooth nonnegative function on the complex plane such that $\sigma(0) = 0$. Then, there exists some point $a_\infty \in \partial\Omega$ such that $\lim\limits_{t\to+\infty} \psi^{-1}(w \pm it, z) = a_\infty$ for any $(w,z) \in D$.*

On the other hand, R. Greene and S. G. Krantz [8] suggested the following conjecture.

Greene-Krantz Conjecture. *If the automorphism group $Aut(\Omega)$ of a smoothly bounded pseudoconvex domain $\Omega \Subset \mathbb{C}^n$ is noncompact, then any orbit accumulation point is of finite type.*

The main results around this conjecture are due to R. Greene and S. G. Krantz [8], K. T. Kim [11], K. T. Kim and S. G. Krantz [12],[13], H. B. Kang [10], M. Landucci [14], J. Byun and H. Gaussier [6].

Let $P_\infty(\partial\Omega)$ be the set of all points in $\partial\Omega$ of infinite type. In [14], M. Landucci proved that the automorphism group of a domain is compact

if $P_\infty(\partial\Omega)$ is a closed interval on the real "normal" line in a complex space with dimension 2. In [6], J. Byun and H. Gaussier also proved that there is no parabolic boundary point if $P_\infty(\partial\Omega)$ is a closed interval transerval to the complex tangent space at one boundary point. For the case which $P_\infty(\partial\Omega)$ is a closed curve on the boundary, is not there exist any parabolic boundary point? In [10], H. B. Kang showed that the automorphism group of the bounded domain $\Omega = \{(z,w) \in \mathbb{C}^2 : |z|^2 + P(w) < 1\}$ is compact, where the function $P(w)$ is smooth and vanishes to infinite order at $w = 0$. In [13], K. T. Kim and S. G. Kantz considered the pseudoconvex domain $\Omega \subset \mathbb{C}^2$ where the local defining function of Ω in a neighborhood of the point of infinite type $(0,0)$ takes the form $\rho(z) = \mathrm{Re} z_1 + \psi(z_2, \mathrm{Im} z_1)$. They pointed out that the origin is not a parabolic boundary point (see [13, Theorem 4.1]). Their proof based on the "fact" that the function ψ vanishes to infinite order at $(0,0)$. But, in general, it is not true, e.g., $\psi(z_2, \mathrm{Im} z_1) = e^{-1/|z_2|^2} + |z_2|^4.|\mathrm{Im} z_1|^2$.

The second aim of this paper is to prove the following theorem which shows that there is no parabolic boundary point of infinite type if $P_\infty(\partial\Omega)$ is a closed curve.

Theorem 1.2. *Let $\Omega \subset \mathbb{C}^2$ be a bounded, pseudoconvex domain in $\mathbb{C}^2$ and $0 \in \partial\Omega$. Assume that*

(1) *$\partial\Omega$ is $\mathcal{C}^\infty$- smooth satisfying Bell's condition R,*
(2) *There exists a neighborhood U of $0 \in \partial\Omega$ such that*

$$\Omega \cap U = \{(z_1, z_2) \in \mathbb{C}^2 : \rho = Rez_1 + P(z_2) + Q(z_2, Imz_1) < 0\},$$

where P and Q satisfy the following comditions

(i) *P is smooth, subharmonic and strictly positive at all points different from the origin, where it vanishes at any order, i.e., $\lim\limits_{z_2 \to 0} \dfrac{P(z_2)}{|z_2|^N} = 0$, $\forall N \geq 0$,*
(ii) *$Q(z_2, Imz_1)$ is smooth and can be written as $Q(z_2, Imz_1) = |z_2|^4 |Im|^2 R(z_2, Imz_1)$ with some smooth function $R(z_2, Imz_1)$.*

Then, $(0,0)$ is not a parabolic boundary point.

Remark 1. By a simple computation, we see that $(0,0)$ is of infinite type, $(it,0)$ with t small enough, are of type greater than or equal to 4 and the other boundary points in a neighborhood of the origin, are strictly pseudoconvex.

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2. Existence of the parabolic boundary point

In this section, we prove Theorem 1.1. To do this, first of all, we recall some notations and some definitions.

For two domains D, Ω in $\mathbb{C}^n$, denote by $Hol(D,\Omega)$ the set of all holomorphic maps from D into Ω. Moreover, we donote by $d(z,\Omega)$ the distance from the point $z \in \Omega$ to $\partial\Omega$ and by Δ the open unit disk in the complex plane.

Definition 2.1. Let p, q be two points in a domain Ω in $\mathbb{C}^n$ and X a vector in $\mathbb{C}^n$.

(a) The **Kobayashi infinitesimal pseudometric** $F_\Omega(p,X)$ is defined by

$$F_\Omega(p,X) = \inf\{\alpha > 0 | \exists g \in Hol(\Delta,\Omega), g(0) = 0, g'(0) = X/\alpha\}$$

(b) The **Kobayashi pseudodistance** $k_\Omega(p,q)$ is defined by

$$k_\Omega(p,q) = \inf \int_a^b F_\Omega(\gamma(t), \gamma'(t))dt,$$

where the infimum is taken over all differentiable curves $\gamma : [a,b] \to \Omega$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

Before proceeding to prove the Theorem 1.1, we need some following lemmas.

Lemma 2.1. *Let Ω be a $\mathcal{C}^1$-smooth, bounded, strictly geometrically convex domain in $\mathbb{C}^2$. Then, there exists $\epsilon_0 > 0$ such that for any $\eta \in \partial\Omega$ and for any $\epsilon \in (0, \epsilon_0]$, there exists a constant $K(\epsilon) > 0$ such*

that the following holds.

$$k_\Omega(z,w) \geq -\frac{1}{2}\ln d(z,\partial\Omega) - K(\epsilon)$$

for any $z, w \in \Omega$ *with* $|z-\eta| < \epsilon,\ |w-\eta| > 3\epsilon$.

Proof. Since $\partial\Omega$ is strictly geometrically convex, there exists a family of holomorphic peak functions

$$\begin{aligned} F : \Omega \times \partial\Omega &\to \mathbb{C} \\ (z,\eta) &\mapsto F(z,\eta) \end{aligned}$$

such that

(i) F is continuous and $F(.,\eta)$ is holomorphic;
(ii) $|F| < 1$;
(iii) There exist a positive constant A and a positive constant ϵ_0 such that $|1 - F(\eta + t\vec{n}_\eta, \eta)| \leq At$ for $t \in [0, \epsilon_0]$, where $\vec{n}_\eta$ is the normal to $\partial\Omega$ at η.

Taking $\epsilon_0 > 0$ small enough, we may assume that $\partial B(\eta, 3\epsilon) \cap \partial\Omega \neq \emptyset$ for $\epsilon \leq \epsilon_0$ and for any $\eta \in \partial\Omega$.

Let γ be some smooth part in Ω such that $\gamma(0) = z,\ \gamma(1) = w$ and $\int\limits_0^1 F_\Omega[\gamma(t), \gamma'(t)]dt \leq k_\Omega(z,w) + 1$. Let $z_0 \in \gamma$ such that $|z_0 - \eta| = 3\epsilon$. We have

$$k_\Omega(z,w) \geq \int_0^1 F_\Omega[\gamma(t),\gamma'(t)]dt - 1 \geq k_\Omega(z,z_0) - 1. \tag{2.2}$$

Let $\tilde\eta \in \partial\Omega$ be such that $z = \tilde\eta + t\vec{n}_\eta$, $t > 0$. We set $u_0 := F(z_0, \tilde\eta)$ and $u := F(z, \tilde\eta)$, u and u_0 are in the unit disk Δ. Then we have

$$k_\Omega(z,z_0) \geq k_\Delta(u,u_0) = \frac{1}{2}\ln\frac{1+|\tau_{u_0}(u)|}{1-|\tau_{u_0}(u)|} \geq -\frac{1}{2}\ln(1-|\tau_{u_0}(u)|), \tag{2.3}$$

where $\tau_{u_0}(u) = \dfrac{u-u_0}{1-\bar u_0 u}$. One easily checks that

$$1 - |\tau_{u_0}(u)| \leq \frac{2|1-u|}{1-|u_0|}. \tag{2.4}$$

Using the properties of F we have

$$|1-u| = |1-F(z,\tilde{\eta})| \leq At = Ad(z, b\Omega). \tag{2.5}$$

Since $|\eta - \tilde{\eta}| \leq |\eta - z| + |z - \tilde{\eta}| < 2\epsilon$ and $|z_0 - \eta| = 3\epsilon$, we have $|z_0 - \tilde{\eta}| \leq \epsilon$.

Let $M(\epsilon) = \sup\limits_{\substack{\eta\in\partial\Omega, z\in\Omega \\ |z-\eta|\geq\epsilon}} |F(z,\eta)|$, $M(\epsilon) < 1$ and therefore

$$1 - |u_0| = 1 - |F(z_0, \tilde{\eta})| \geq 1 - M(\epsilon) > 0. \tag{2.6}$$

From (2.4),(2.5) and (2.6) we get

$$1 - |\tau_{u_0}(u)| \leq \frac{2A}{1-M(\epsilon)} d(z, \partial\Omega). \tag{2.7}$$

Then form (2.2),(2.3) and (2.7) we obtain

$$k_\Omega(z,w) \geq -\frac{1}{2}\ln d(z,\partial\Omega) - \frac{1}{2}\ln\frac{2A}{1-M(\epsilon)} - 1 \tag{2.8}$$

and this completes the proof. □

Lemma 2.2. *Let Ω be a $\mathcal{C}^1$-smooth, bounded, strictly geometrically convex domain in $\mathbb{C}^2$ and let $\eta, \eta' \in \partial\Omega$ with $\eta \neq \eta'$. Then there exist $\epsilon > 0$ and a constant K suth that*

$$k_\Omega(z,w) \geq -\frac{1}{2}\ln d(z,\partial\Omega) - \frac{1}{2}\ln d(w,\partial\Omega) - K,$$

for any $z \in B(\eta, \epsilon)$ and any $w \in B(\eta', \epsilon)$.

Proof. Let η and η' be two distinct points on $\partial\Omega$. Suppose that $|z-\eta| < \epsilon$ and $|w - \eta'| < \epsilon$ and let γ be a C^1 part in Ω connecting z and w such that $k_\Omega(z,w) \geq \int\limits_0^1 F_\Omega[\gamma(t), \gamma'(t)]dt - 1$. If ϵ is small enough we may find $z_0 \in \gamma$ such that $|z_0 - \eta| > 3\epsilon$ and $|z_0 - \eta'| > 3\epsilon$. Let $z_0 = \gamma(t_0)$, then

$$\begin{aligned} k_\Omega(z,w) &\geq \int\limits_0^{t_0} F_\Omega(\gamma(t),\gamma'(t))dt + \int_{t_0}^1 F_\Omega(\gamma(t),\gamma'(t))dt - 1 \\ &\geq k_\Omega(z, z_0) + k_\Omega(z_0, w) - 1 \\ &\geq -\frac{1}{2}\ln d(z,\partial\Omega) - \frac{1}{2}\ln d(w,\partial\Omega) - 2K(\epsilon) - 1, \end{aligned}$$

where the last inequality is obtained by applying two times Lemma 2.1 □

We now recall the difinition of horosphere

Definition 2.2. Let $a \in \Omega,\ \eta \in \partial\Omega,\ R > 0$. The big horosphere with pole a, center η and radius R in Ω is defined as follows

$$F_a^\Omega(\eta, R) = \{z \in \Omega : \liminf_{w \to \eta} \left[k_\Omega(z, w) - k_\Omega(a, w)\right] < \frac{1}{2} \ln R\}.$$

Lemma 2.3. *If Ω is a $\mathcal{C}^1$-smooth, bounded, strictly geometrically convex domain in $\mathbb{C}^2$, then $\overline{F_a^\Omega(\eta, R)} \cap \partial\Omega \subset \{\eta\}$ for any $a \in \Omega,\ \eta \in \partial\Omega,\ R > 0$.*

Proof. If there exists $\eta' \in \partial\Omega \cap \overline{F_a^\Omega(\eta, R)}$ then we can find a sequence $\{z_n\} \subset \Omega$ with $z_n \to \eta'$ and a sequence $\{w_n\} \subset \Omega$ with $z_n \to \eta$ such that

$$k_\Omega(z_n, w_n) - k_\Omega(a, w_n) < \frac{1}{2} \ln R \tag{2.9}$$

By Lemma 2.2, the following estimate occurs if $\eta \neq \eta'$ and n great enough.

$$k_\Omega(z_n, w_n) \geq -\frac{1}{2} \ln d(z_n, \partial\Omega) - \frac{1}{2} \ln d(w_n, \partial\Omega) - K, \tag{2.10}$$

where K is a constant.

On the other hand, we have

$$k_\Omega(a, w_n) \leq -\frac{1}{2} \ln d(w_n, \partial\Omega) + K(a), \tag{2.11}$$

since $\partial\Omega$ is smooth.

From (2.9), (2.10) and (2.11) we get

$$-\frac{1}{2} \ln d(z_n, \partial\Omega) \lesssim 1, \tag{2.12}$$

which is absurd. □

Proof of Theorem 1.1. Set $a_n := \psi^{-1}(-t_n, 0)$ where $\lim t_n = +\infty$, after taking a subsequence we may assume that $\lim a_n = a_\infty \in \partial\Omega$. We may also assume that a_∞ is the origin in $\mathbb{C}^2$.

Set $b_t := \psi^{-1}(-1+it, 0)$, according to Lemma 2.3, it suffices to show that there exists $R_0 > 0$ such that

$$\{b_t : t \in \mathbb{R}\} \subset F^{\Omega}_{a_0}(a_\infty, R_0). \tag{2.13}$$

Since $a_n \to a_\infty$, we have

$$\liminf_{w\to a_\infty} \big[k_\Omega(b_t, w) - k_\Omega(a_0, w)\big] \leq \liminf_{n\to+\infty} \big[k_\Omega(b_t, a_n) - k_\Omega(a_0, a_n)\big]. \tag{2.14}$$

Then by the invariance of the Kobayashi metric and the convexity of D we have

$$\begin{aligned} k_\Omega(b_t, a_n) - k_\Omega(a_0, a_n) &= k_D[(-1+it, 0), (-t_n, 0)] - k_D[(-t_0, 0), (-t_n, 0)] \\ &= k_H(-1+it, -t_n) - k_H(-t_0, -t_n), \end{aligned} \tag{2.15}$$

where H is the left half plane $\{\mathrm{Re} w < 0\}$.

Let $\sigma : \mathrm{H} \to \Delta$ be a biholomorphism between H and the disk Δ given by $\sigma(w) = \dfrac{w+1}{w-1}$. Set $z_t := \sigma(-1+it) = \dfrac{it}{-2+it}$ and $x_n := \sigma(-t_n) = \dfrac{-t_n+1}{-t_n-1}$. Then we have

$$\begin{aligned} &k_{\mathrm{H}}(-1+it, -t_n) - k_{\mathrm{H}}(-t_0, -t_n) = k_\Delta(z_t, x_n) - k_\Delta(x_0, x_n) \\ &= \ln\Big(\frac{|1-x_nz_t| + |x_n - z_t|}{|1-x_nz_t| - |x_n - z_t|}.\frac{|1-x_nx_0| + |x_n - x_0|}{|1-x_nx_0| - |x_n - x_0|}\Big) \\ &= \ln\Big(\frac{|1-x_nx_0| + |x_n - x_0|}{|1-x_nz_t| - |x_n - z_t|}.\frac{|1-x_nz_t| + |x_n - z_t|}{|1-x_nx_0| - |x_n - x_0|}\Big) \\ &= \ln\Big(\frac{|1-x_nx_0|^2 - |x_n - x_0|^2}{|1-x_nz_t|^2 - |x_n - z_t|^2}.\Big[\frac{|1-x_nz_t| + |x_n - z_t|}{|1-x_nx_0| - |x_n - x_0|}\Big]^2\Big) \\ &= \ln\Big(\frac{1-x_0^2}{1-|z_t|^2}.\Big[\frac{|1-x_nz_t| + |x_n - z_t|}{|1-x_nx_0| - |x_n - x_0|}\Big]^2\Big). \end{aligned} \tag{2.16}$$

From (2.15) and (2.16)we have

$$\begin{aligned} \lim_{n\to\infty} \big[k_\Omega(b_t, a_n) - k_\Omega(a_0, a_n)\big] &= \ln\Big(\frac{1-x_0^2}{1-|z_t|^2}.\frac{|1-z_t|^2}{|1-x_0|^2}\Big) \\ &= \ln\frac{1-x_0^2}{|1-x_0|^2}. \end{aligned} \tag{2.17}$$

Finally, (2.13) follows directly from (2.14) and (2.17) when $\ln\frac{1-x_0^2}{|1-x_0|^2} < \frac{1}{2}\ln R_o$. $\square$

3. NON-EXISTENCE OF THE PARABOLIC BOUNDARY POINT OF INFINITE TYPE

Let Ω be a domain satisfying conditions given in Theorem 1.2. In this section, the non-existence of the parabolic boundary point of infinite type of Ω is proved. First of all, we need some following lemmas.

Lemma 3.1. *There do not exist $a, b \in \mathbb{C}$ with $Rea \neq 0$ and $b \neq 0$ such that*

$$Re[aP(z) + bz^k P'(z)] = \gamma(z)P(z), \tag{3.18}$$

for some $k \in \mathbb{N},\ k > 1$ and for every $|z| < \epsilon_0$ with $\epsilon_0 > 0$ small enough, where $\gamma(z)$ is smooth and $\gamma(z) \to 0$ as $z \to 0$.

Proof. Suppose that there exist $a, b \in \mathbb{C}$ with $\mathrm{Re}a \neq 0$ and $b \neq 0$ such that

$$\mathrm{Re}[aP(z) + bz^k P'(z)] = \gamma(z)P(z), \tag{3.19}$$

for some $k \in \mathbb{N},\ k > 1$ and for every $|z| < \epsilon_0$ with $\epsilon_0 > 0$ small enough. This equation is equivalent to

$$1 + \mathrm{Re}[\frac{b}{\mathrm{Re}a} z^k \frac{P'(z)}{P(z)}] = \gamma_1(z), \quad \forall\, 0 < |z| < \epsilon_0, \tag{3.20}$$

where $\gamma_1(z) = \gamma(z)/\mathrm{Re}a$. Let $F(z) = \ln P(z)$ and write $z = re^{i\varphi}$, $\dfrac{b}{2\mathrm{Re}a} = \dfrac{1}{R}e^{i\psi}$. Then, by (3.20), we get

$$\frac{\partial F}{\partial x}(z)\cos(k\varphi + \psi) + \frac{\partial F}{\partial y}(z)\sin(k\varphi + \psi) = -\frac{R}{r^k} + \frac{R}{r^k}\gamma_1(z).$$

If we set $\varphi_0 = \dfrac{2\pi - \psi}{k-1}$, then

$$\frac{\partial F}{\partial x}(re^{i\varphi_0})\cos(\varphi_0) + \frac{\partial F}{\partial y}(re^{i\varphi_0})\sin(\varphi_0) = -\frac{R}{r^k} + \frac{R}{r^k}\gamma_1(re^{i\varphi_0}).$$

Let $g(r) := F(re^{i\varphi_0})$. It is easy to see that

$$g'(r) = -\frac{R}{r^k} + \frac{R}{r^k}\gamma_1(re^{i\varphi_0}).$$

Let $h(r) := g(r) + \frac{R}{1-k}\frac{1}{r^k - 1}$. Then

$$h'(r) = \frac{R}{r^k}\gamma_1(re^{i\varphi_0}).$$

We may assume that there exists r_0 small enough such that $|h'(r)| \leq \frac{R}{2r^k}$, for every $0 < r \leq r_0$. Thus, we have the following estimate

$$\begin{aligned}|h(r)| &\leq |h(r_0)| + \Big|\int_{r_0}^{r} |h'(r)|dr\Big|\\ &\leq |h(r_0)| + \frac{R}{2}\Big|\int_{r_0}^{r} r^{-k}dr\Big|\\ &\leq |h(r_0)| - \frac{R}{2(k-1)}r_0^{1-k} + \frac{R}{2(k-1)}r^{1-k}.\end{aligned}$$

Hence,

$$g(r) \geq \frac{R}{k-1}r^{1-k} - |h(r_0)| + \frac{R}{2(k-1)}r_0^{1-k} - \frac{R}{2(k-1)}r^{1-k}.$$

It implies that $\lim_{r\to 0^+} g(r) = +\infty$. This means that $P(re^{i\varphi_0}) \not\to 0$ as $r \to 0^+$. It is impossible. $\square$

Lemma 3.2. *There do not exist $a, b \in \mathbb{C}$ with Rea $\neq 0$ and $b \neq 0$ such that*

$$Re[aP^{n+1}(z) + bz^kP'(z)] = \gamma(z)P^{n+1}(z), \tag{3.21}$$

for some $k \in \mathbb{N},\ k > 1$ and for every $|z| < \epsilon_0$ with $\epsilon_0 > 0$ small enough, where $\gamma(z) \to 0$ as $z \to 0$.

Proof. Suppose that there exist $a, b \in \mathbb{C}$ with Re$a \neq 0$ and $b \neq 0$ such that

$$\mathrm{Re}[aP^{n+1}(z) + bz^kP'(z)] = \gamma(z)P^{n+1}(z), \tag{3.22}$$

for some $k \in \mathbb{N},\ k > 1$ and for every $|z| < \epsilon_0$ with $\epsilon_0 > 0$ small enough. This equation is equivalent to

$$1 + \mathrm{Re}[\frac{b}{\mathrm{Re}a}z^k\frac{P'(z)}{P^{n+1}(z)}] = \gamma_1(z), \quad \forall\, 0 < |z| < \epsilon_0, \tag{3.23}$$

where $\gamma_1(z) = \gamma(z)/\mathrm{Re}a$. Let $F(z) = \dfrac{1}{P^n(z)}$ and write $z = re^{i\varphi}$, $\dfrac{-b}{2n\mathrm{Re}a} = \dfrac{1}{R}e^{i\psi}$. By (3.23), we get

$$\frac{\partial F}{\partial x}(z)\cos(k\varphi+\psi) + \frac{\partial F}{\partial y}(z)\sin(k\varphi+\psi) = -\frac{R}{r^k} + \frac{R}{r^k}\gamma_1(z).$$

If we set $\varphi_0 = \dfrac{2\pi-\psi}{k-1}$, then

$$\frac{\partial F}{\partial x}(re^{i\varphi_0})\cos(\varphi_0) + \frac{\partial F}{\partial y}(re^{i\varphi_0})\sin(\varphi_0) = -\frac{R}{r^k} + \frac{R}{r^k}\gamma_1(re^{i\varphi_0}).$$

Let $g(r) := F(re^{i\varphi_0})$. Then we see that

$$g'(r) = -\frac{R}{r^k} + \frac{R}{r^k}\gamma_1(re^{i\varphi_0}).$$

Let $h(r) := g(r) + \dfrac{R}{1-k}\dfrac{1}{r^k-1}$. Then we may assume that there is r_0 small enough such that

$$|h'(r)| \le \frac{3R}{2r^k},$$

for every $0 < r \le r_0$. Thus, we have the following estimate

$$\begin{aligned}
|g(r)| &\le |g(r_0)| + \Big|\int_{r_0}^{r} |g'(r)|dr\Big| \\
&\le |g(r_0)| + \frac{3R}{2}\Big|\int_{r_0}^{r} r^{-k}dr\Big| \\
&\le |g(r_0)| - \frac{3R}{2(k-1)}r_0^{1-k} + \frac{3R}{2(k-1)}r^{1-k}.
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{P^n(re^{i\varphi_0})} \lesssim \frac{1}{r^{1-k}},$$
$$P(re^{i\varphi_0}) \gtrsim r^{\frac{k-1}{n}}.$$

This means that $P(re^{i\varphi_0})$ does not vanish to infinite order at $r = 0$. It is a contradiction. $\square$

Lemma 3.3. *There do not exist $a, b \in \mathbb{C}$ with $Rea \neq 0$ and $b \neq 0$ such that*

$$Re[aP^{n+1}(z) + bzP'(z)] = \gamma(z)P^{n+1}(z), \tag{3.24}$$

for some $n \geq 0$ and for every $|z| < \epsilon_0$ with $\epsilon_0 > 0$ small enough, where $\gamma(z) \to 0$ as $z \to 0$.

Proof. Suppose that there exist $a, b \in \mathbb{C}$ with $\mathrm{Re}a \neq 0$ and $b \neq 0$ such that (3.24) holds. We first consider the case $n = 0$. Then the equation (3.24) is equivalent to

$$\mathrm{Re}[\frac{b}{\mathrm{Re}a} z \frac{\partial}{\partial z} \ln P(z)] = -1 + \gamma_1(z), \tag{3.25}$$

where $\gamma_1(z) := \gamma(z)/\mathrm{Re}a$. Let $u(z) := \ln P(z)$ and write $\frac{b}{2\mathrm{Re}a} = \alpha + i\beta,\ z = x + iy$. Then, by (3.25), we have the following first order partial differential equation

$$(\alpha x - \beta y)\frac{\partial}{\partial x} u(x, y) + (\beta x + \alpha y)\frac{\partial}{\partial y} u(x, y) = -1 + \gamma_1(x, y). \tag{3.26}$$

In order to solve this partial differential equation, we need to solve the following system of differential equation.

$$\begin{cases} x'(t) = \alpha x - \beta y \\ y'(t) = \beta x + \alpha y,\ t \in \mathbb{R}. \end{cases}$$

By a simple computation, we obtain

$$\begin{cases} x(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) \\ y(t) = -c_2 e^{\alpha t}\cos(\beta t) + c_1 e^{\alpha t}\sin(\beta t,\ t \in \mathbb{R}, \end{cases} \tag{3.27}$$

where c_1, c_2 are two constant real numbers. Let $g(t) := u(x(t), y(t))$. Then $g'(t) = -1 + \gamma_1(x(t), y(t))$. Thus, $g(t) = -t + \int\limits_{t_0}^{t} \gamma_1(x(s), y(s))ds + t_0 + g(t_0)$. From (3.27), we get

$$x^2 + y^2 = (c_1^2 + c_2^2)e^{2\alpha t},\ t \in \mathbb{R}. \tag{3.28}$$

Consider three following cases

Case 1. $\alpha = 0$. In this case, take $c_1 = r > 0,\ c_2 = 0$, where r small enough. Then, on each small circle $\{x(t) = r\cos(t),\ y(t) = r\sin(t),\ t \in [0, 2\pi]\}$, $g(t) = -t + \int\limits_{0}^{t} \gamma_1(x(s), y(s))ds + u(r, 0)$. Taking r small enough, we may assume that $|\gamma_1(x(s), y(s))| \leq 1/2$ for all $s \in [0, 2\pi]$. It is easy

to see that $|g(2\pi) - g(0)| \geq \pi$. This is absurd since $g(2\pi) = g(0) = u(r, 0)$.

Case 2. $\alpha > 0$. By (3.28), $(x(t), y(t)) \to 0$ as $t \to -\infty$. Then, $u(x(t), y(t)) \to +\infty$ as $t \to -\infty$. This is a contradiction.

Case 3. $\alpha < 0$. By (3.28), we have $(x(t), y(t)) \to 0$ as $t \to +\infty$ and $t = \frac{1}{2\alpha} \ln \frac{x^2+y^2}{c_1^2+c_2^2}$. Taking $t_0 > 0$ big enough, we may assume that $|\gamma_1(x(s), y(s))| \leq 1$ for all $s \geq t_0$. Then for all $t \geq t_0$, we have

$$
\begin{aligned}
g(t) &\geq -(t-t_0) - \left|\int_{t_0}^{t} \gamma_1(x(s), y(s))ds\right| - |g(t_0)| \\
&\geq -(t-t_0) - \left|\int_{t_0}^{t} |\gamma_1(x(s), y(s))|ds\right| - |g(t_0)| \\
&\geq -(t-t_0) - |t-t_0| - |g(t_0)| \\
&\geq -2(t-t_0) - |g(t_0)|.
\end{aligned}
$$

Hence, for all $t \geq t_0$, we obtain

$$
\begin{aligned}
P(z(t)) &\gtrsim e^{-2t} \\
&\gtrsim |z(t)|^{-1/\alpha},
\end{aligned}
$$

where $z(t) := x(t) + iy(t)$. It is impossible since P vanishes to infinite order at 0.

We now consider the case $n > 0$. Then the equation (3.24) is equivalent to

$$
\mathrm{Re}\left[\frac{b}{-n\mathrm{Re}a} z \frac{\partial}{\partial z} \frac{1}{P^n(z)}\right] = -1 + \gamma_1(z), \tag{3.29}
$$

where $\gamma_1(z) := \gamma(z)/\mathrm{Re}a$. Let $u(z) := \frac{1}{P^n(z)}$ and write $\frac{b}{-2n\mathrm{Re}a} = \alpha + i\beta$, $z = x + iy$. Then, by (3.29), we have the following first order partial differential equation

$$
(\alpha x - \beta y)\frac{\partial}{\partial x} u(x,y) + (\beta x + \alpha y)\frac{\partial}{\partial y} u(x,y) = -1 + \gamma_1(x,y). \tag{3.30}
$$

In order to solve this partial differential equation, we need to solve the following system of differential equation.

$$\begin{cases} x'(t) = \alpha x - \beta y \\ y'(t) = \beta x + \alpha y,\, t \in \mathbb{R}. \end{cases}$$

By a simple computation, we obtain

$$\begin{cases} x(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) \\ y(t) = -c_2 e^{\alpha t}\cos(\beta t) + c_1 e^{\alpha t}\sin(\beta t),\ t \in \mathbb{R}, \end{cases} \tag{3.31}$$

where c_1, c_2 are two constant real numbers. Let $g(t) := u(x(t), y(t))$. Then $g'(t) = -1 + \gamma_1(x(t), y(t))$. Thus, $g(t) = -t + \int\limits_{t_0}^{t} \gamma_1(x(s), y(s))ds + t_0 + g(t_0)$. From (3.31), we get

$$x^2 + y^2 = (c_1^2 + c_2^2)e^{2\alpha t},\ t \in \mathbb{R}. \tag{3.32}$$

Consider three following cases

Case 1. $\alpha = 0$. In this case, take $c_1 = r > 0,\ c_2 = 0$, where r small enough. Then, on each small circle $\{x(t) = r\cos(t),\ y(t) = r\sin(t),\ t \in [0, 2\pi]\}$, $g(t) = -t + \int\limits_{0}^{t} \gamma_1(x(s), y(s))ds + u(r, 0)$. Taking r small enough, we may assume that $|\gamma_1(x(s), y(s))| \leq 1/2$ for all $s \in [0, 2\pi]$. It is easy to see that $|g(2\pi) - g(0)| \geq \pi$. This is not possible since $g(2\pi) = g(0) = u(r, 0)$.

Case 2. $\alpha < 0$. By (3.32), $(x(t), y(t)) \to 0$ as $t \to +\infty$. Then, $u(x(t), y(t)) \to -\infty$ as $t \to -\infty$. It is a contradiction.

Case 3. $\alpha > 0$. By (3.32), we have $(x(t), y(t)) \to 0$ as $t \to -\infty$ and $t = \dfrac{1}{2\alpha}\ln\dfrac{x^2+y^2}{c_1^2+c_2^2}$. Taking $t_0 < 0$ such that $|t_0|$ big enough, we may assume that $|\gamma_1(x(s), y(s))| \leq 1$ for all $s \leq t_0$. Then for all $t \leq t_0$, we

have the following estimate

$$\begin{aligned}
g(t) &\le -(t-t_0) + \left|\int_{t_0}^{t} \gamma_1(x(s), y(s))ds\right| + |g(t_0)| \\
&\le -(t-t_0) + \left|\int_{t_0}^{t} |\gamma_1(x(s), y(s))|ds\right| + |g(t_0)| \\
&\le -(t-t_0) + |t-t_0| + |g(t_0)| \\
&\le -2(t-t_0) + |g(t_0)|.
\end{aligned}$$

Hence, for all $t \le t_0$, we obtain

$$\begin{aligned}
P^n(z(t)) &\gtrsim \frac{1}{-2t} \\
&\gtrsim \frac{-1}{\ln|z(t)|},
\end{aligned}$$

where $z(t) := x(t) + iy(t)$. This implies that

$$\lim_{t\to-\infty} \frac{P(z(t))}{|z(t)|} = +\infty.$$

This is impossible since P vanishes to infinite order at 0. $\square$

Let $F = (f, g) \in Aut(\Omega)$ be such that $F(0,0) = (0,0)$. Because of Bell's condition R of $\partial\Omega$, F extends smoothly to the boumdary of Ω. Let U be a a neighborhood of $(0,0)$. Then, there exists a neighborhood V of $(0,0)$ such that

$$F(\overline{\Omega} \cap V) \subset \overline{\Omega} \cap U. \tag{3.33}$$

The following lemma is similar to Lemma 2.5 of [14].

Lemma 3.4. *Let $F = (f, g) \in Aut(\Omega)$. Let U, V be two neighborhoods of $(0,0)$ such that (3.33) holds. Then, for any $(z_1, z_2) \in V$,*

(i) $g(z_1, 0) = 0$.
(ii) $f(z_1, z_2) = f(z_2)$

Proof. a) Let U, V be two neighborhoods of $(0,0)$ such that (3.33) holds. Let γ be the set of all points $(it, 0) \in \partial\Omega \cap U$. By Bell's condition R, the restriction to $\partial\Omega$ of the extension of F to $\overline{\Omega}$ defines a C-R automorphism of $\partial\Omega$. Since the D'Angelo type is a C-R invariant, we

have $F(\gamma \cap V) \subset \gamma$. Hence, $g(it, 0) = 0$ and $\mathrm{Re} f(it, 0) = 0$. Since $h(z_1) := g(z_1, 0) \in Hol(\mathrm{H}) \cap \mathcal{C}^\infty(\overline{\mathrm{H}})$, $g(z_1, 0) \equiv 0$. Here, we denote H by $\mathrm{H} = \{z_1 \in \mathbb{C} : \mathrm{Re} z_1 < 0\}$.

b) A classical argument based on the Hopf's lemma shows that $(\rho \circ F)(z_1, z_2)$ is also a defining function on V. In particular, there exists a smooth function $k(z_1, z_2)$ which is strictly positive and such that, for any $(z_1, z_2) \in V$,

$$\begin{aligned} &\mathrm{Re} z_1 + P(z_2) + Q(z_2, \mathrm{Im} z_1) \\ &\qquad = k(z_1, z_2)\big[\mathrm{Re} f(z_1, z_2) + P(g(z_1, z_2)) + Q(g(z_1, z_2), \mathrm{Im} f(z_1, z_2)\big]. \end{aligned} \tag{3.34}$$

We claim that for any $N \geq 1$ and any $(it, 0) \in \gamma \cap V$

$$\frac{\partial^N}{\partial z_2^N}\Big(\mathrm{Re} f(z_1, z_2) + P(g(z_1, z_2)) + Q(g(z_1, z_2), \mathrm{Im} f(z_1, z_2))\Big)\Big|_{(it,0)} = 0. \tag{3.35}$$

In fact, for any $(it, 0) \in \gamma \cap V$ we have that

$$\mathrm{Re} f(it, 0) + P(g(it, 0)) + Q(g(it, 0), \mathrm{Im} f(it, 0)) = 0.$$

From (3.34), it follows that

$$\frac{\partial}{\partial z_2}\Big(\mathrm{Re} f(z_1, z_2) + P(g(z_1, z_2)) + Q(g(z_1, z_2), \mathrm{Im} f(z_1, z_2))\Big)\Big|_{(it,0)} = 0,$$

which implies (3.35) for $N = 1$. Taking the N-th derivative with respect to z_2 of (3.34) and using an inductive argument, it follows that (3.35) holds also for any $N > 1$. From a), (3.35) and the property $(2.i)$ we get that for any $N \geq 1$ and any $(it, 0) \in \partial\Omega \cap V$

$$\frac{\partial^N}{\partial z_2^N} f(it, 0) = 0. \tag{3.36}$$

Using the same arguments as for (a), we see that (3.36) implies (b). $\square$

Proof of theorem 1.2. Suppose that $(0, 0) \in \partial\Omega$ be a parabolic boundary point associated with a one-parameter group $\{F_\theta\}_{\theta \in \mathbb{R}} \subset Aut(\Omega)$. Let H be the vector field generating the group $\{F_\theta\}_{\theta \in \mathbb{R}}$, i.e.,

$$H(z) = \frac{d}{d\theta} F_\theta(z)\Big|_{\theta=0}.$$

Since Ω satisfies Bell's condition R, each automorphism of Ω extends to be of class $\mathcal{C}^\infty$ on $\overline{\Omega}$. Therefore, $H \in Hol(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$. Furthermore, since $F_\theta(\partial\Omega) \subset \partial\Omega$, it follows that $H(z) \in T_z(\partial\Omega)$ for all $z \in \partial\Omega$, i.e.,

$$(\mathrm{Re} H)\rho(\zeta) = 0, \ \forall\, \zeta \in \partial\Omega. \tag{3.37}$$

A vector field $H \in Hol(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$ satisfying (3.37) is called to be a holomorphic tangent vector field for domain Ω. Since $F_\theta(0,0) = (0,0)$, it follows from Lemma 3.4 that $F_\theta(z_1, z_2) = (f_\theta(z_1), z_2 g_\theta(z_1, z_2))$, where f_θ and g_θ are holomorphic on $U \cap \Omega$, where U is a neighborhood of $(0,0)$. Hence, the vector field H has the form

$$H(z_1, z_2) = h_1(z_1)\frac{\partial}{\partial z_1} + z_2 h_2(z_1, z_2)\frac{\partial}{\partial z_2},$$

where h_1 and h_2 is holomophic on Ω and, is of class $\mathcal{C}^\infty$ up to the boundary $\partial\Omega$. Moreover, h_1 vanishes at the origin. By a simple computation, we get

$$\begin{aligned}\frac{\partial}{\partial z_1}\rho(z_1, z_2) &= \frac{1}{2} + \frac{\partial}{\partial z_1}Q(z_2, \mathrm{Im} z_1),\\ \frac{\partial}{\partial z_2}\rho(z_1, z_2) &= P'(z_2) + \frac{\partial}{\partial z_2}Q(z_2, \mathrm{Im} z_1).\end{aligned}$$

Since $H(z)$ is a tangent vector field to $\partial\Omega$, we have

$$\begin{aligned}\mathrm{Re}\Big[&(\frac{1}{2} + \frac{\partial}{\partial z_1}Q(z_2, \mathrm{Im} z_1))h_1(z_1) + \\ &+ (P'(z_2) + \frac{\partial}{\partial z_2}Q(z_2, \mathrm{Im} z_1))z_2 h_2(z_1, z_2)\Big] = 0,\end{aligned} \tag{3.38}$$

for all $(z_1, z_2) \in \partial\Omega$. For any $(it, 0) \in \partial\Omega \cap U$, we have

$$\mathrm{Re} h_1(it) = 0. \tag{3.39}$$

Since $h_1 \in Hol(\mathrm{H}) \cap \mathcal{C}^\infty(\overline{\mathrm{H}})$, where H is the left half-plane, by the Schwarz reflection principle, h_1 can be extended to be a holomorphic on a neighborhood of $z_1 = 0$. From (3.38), it follows that, for any $(-P(z_2), z_2) \in \partial\Omega \cap U$,

$$\mathrm{Re}\Big[\frac{1}{2}h_1(-P(z_2)) + z_2 P'(z_2) h_2(-P(z_2), z_2)\Big] = 0. \tag{3.40}$$

Expanding h_1 and h_2 into Taylor series about the origin, we get $h_1(z_1) = \sum\limits_{n=0}^{\infty} a_n z_1^n$ and $h_2(z_1,z_2) = \sum\limits_{k=0}^{\infty} b_k(z_1) z_2^k$, where $a_n \in \mathbb{C}$, $b_k \in Hol(\mathrm{H}) \cap \mathcal{C}^\infty(\overline{\mathrm{H}})$, for any $n,k \in \mathbb{N}$. Note that $a_0 = 0$ since $h_1(0) = 0$. If there exists an integer number $n \geq 1$ such that $\mathrm{Re} a_n \neq 0$, then the biggest term in $\mathrm{Re}[\frac{1}{2} h_1(-P(z_2))]$ has the form $\mathrm{Re} a_n P^n(z_2)$. Therefore, there exists at least $k \in \mathbb{N}$ such that either $b_k(0) \neq 0$ or $b_k(z_1)$ vanishes to finite order at $z_1 = 0$. Then the biggest term in $\mathrm{Re}\big[z_2 P'(z_2) h_2(-P(z_2), z_2)\big]$ has the form $\mathrm{Re}\big[b z_2^k P'(z_2) P^l(z_2)\big]$, where $b \in \mathbb{C}^*$, $l \in \mathbb{N}$. By (3.40), there exists $\epsilon_0 > 0$ such that

$$\mathrm{Re}\big[a_n P^{n-l}(z_2) + b z_2^k P'(z_2)\big] = o(P^{n-l}(z_2)), \tag{3.41}$$

for all $|z_2| < \epsilon_0$. It is easy to see that $n > l$. Thus, by Lemma 3.1, Lemma 3.2, and Lemma 3.3, we get $\mathrm{Re} a_n = b = 0$. This is a contradiction. Therefore, $\mathrm{Re} a_n = 0$ for every $n \geq 1$ and thus, we can write $h_1(z_1) = i. \sum\limits_{n=1}^{\infty} \alpha_n z_1^n$, where $\alpha_n \in \mathbb{R}$, $n = 1, 2, \cdots$. Let $u(z_1) := \mathrm{Re} h_1(z_1)$. Then the function u is harmonic on the left haft-plane H and is smooth up to the boundary ∂H. By (3.39), we have, for any real number t small enough, $u(it) = 0$. Moreover, $u(-t) = 0$ for any t small enough since $h_1(z_1) = i \sum\limits_{n=1}^{\infty} \alpha_n z_1^n$. Hence, by the maximum principle, we conclude that $u(z_1) \equiv 0$. Consequensely, $h_1(z_1) \equiv 0$ and hence, H becomes a planar vector field. This is impossible since $\partial\Omega$ is not flat near the origin. So the proof is completed. $\square$

References

[1] E. Bedford and S. Pinchuk, *Domains in $\mathbb{C}^2$ with noncompact groups of automorphisms,* Math. USSR Sbornik 63(1989), 141-151.

[2] E. Bedford and S. Pinchuk, *Domains in $\mathbb{C}^{n+1}$ with noncompact automorphism group,* J. Geom. Anal. 1 (1991), 165-191.

[3] E. Bedford and S. Pinchuk, *Domains in $\mathbb{C}^2$ with noncompact automorphism groups,* Indiana Univ. Math. Journal 47(1998), 199-222.

[4] S. Bell, *Compactness of families of holomorphic mappings up to the boundary* Lecture Notes in Math. Vol. 1268, Springer-Verlag, Berlin/New Jork, 1987, 29-42.

[5] F. Berteloot, *Characterization of models in* $\mathbb{C}^2$ *by their automorphism groups,* Internat. J. Math. 5(1994), 619-634.

[6] J. Byun and H. Gaussier, *On the compactness of the automorphism group of a domain*, C. R. Acad. Sci. Paris, Ser. 1341 (2005), 545-548.

[7] J. P. D'Angelo, *Real hypersurfaces, orders of contact, and applications,* Ann. Math. 115 (1982), 615-637.

[8] R. Greene and S. G. Krantz, *Techniques for studying automorphisms of weakly pseudoconvex domains,* Math. Notes, Vol 38, Princeton Univ. Press, Princeton, NJ, 1993, 389-410.

[9] A. Isaev and S. G. Krantz, *Domains with non-compact automorphism group : A survey,* Adv. Math. 146 (1999), 1-38.

[10] H. B. Kang, *Holomorphic automorphisms of certain class of domains of infinite type,* Tohoku Math. J. 46 (1994), 345-422.

[11] K. T. Kim, *On a boundary point repelling automorphism orbits,* J. Math. Anal. Appl. 179 (1993), 463-482.

[12] K. T. Kim and S. G. Krantz, *Convex scaling and domains with non-compact automorphism group,* Illinois J. Math. 45 (2001), 1273-1299.

[13] K. T. Kim and S. G. Krantz, *Some new results on domains in complex space with non-compact automorphism group,* J. Math. Anal. Appl. 281 (2003), 417-424.

[14] M. Landucci, *The automorphism group of domains with boundary points of infinite type,* Illinois J. Math. 48 (2004), 33-40.

[15] B. Wong, *Characterization of the ball in* $\mathbb{C}^n$ *by its automorphism group*, Invent. Math. 41 (1977), 253-257.

[16] J. P. Rosay, *Sur une caracterisation de la boule parmi les domaines de* $\mathbb{C}^n$ *par son groupe* d'*automorphismes*, Ann. Inst. Fourier 29 (4) (1979), 91-97.

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